Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of Sound and Vibration 274 (2004) 701–724
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

Optimal feedback control of strongly non-linear systems excited by bounded noise

W.Q. Zhu[a,*], Z.L. Huang[a], J.M. Ko[b], Y.Q. Ni[b]

[a] *Department of Mechanics, Zhejiang University, Hangzhou 310027, People's Republic of China*
[b] *Department of Civil and Structural Engineering, Hong Kong Polytechnic University, Kowloon, Hong Kong, People's Republic of China*

Received 12 December 2002; accepted 17 June 2003

Abstract

A strategy for non-linear stochastic optimal control of strongly non-linear systems subject to external and/or parametric excitations of bounded noise is proposed. A stochastic averaging procedure for strongly non-linear systems under external and/or parametric excitations of bounded noise is first developed. Then, the dynamical programming equation for non-linear stochastic optimal control of the system is derived from the averaged Itô equations by using the stochastic dynamical programming principle and solved to yield the optimal control law. The Fokker–Planck–Kolmogorov equation associated with the fully completed averaged Itô equations is solved to give the response of optimally controlled system. The application and effectiveness of the proposed control strategy are illustrated with the control of cable vibration in cable-stayed bridges and the feedback stabilization of the cable under parametric excitation of bounded noise.

1. Introduction

As indicated by Housner et al. [1] that stochastic control of non-linear systems is an interesting yet difficult problem. Most studies in the field of stochastic control of structures until recently use a linear quadratic Gaussian (LQG) strategy. In the last few years, a non-linear stochastic optimal control strategy has been proposed by the present first author and his co-workers [2–4] based on the stochastic averaging method for quasi Hamiltonian systems [5–7] and the stochastic dynamical programming principle [8–10]. The proposed control strategy has been extended to feedback

*Corresponding author. Tel.: +86-571-8799-1150; fax: +86-571-8795-2651.
E-mail address: wqzhu@yahoo.com (W.Q. Zhu).

doi:10.1016/S0022-460X(03)00746-6

stabilization [11] and feedback maximization of reliability [12,13] of non-linear stochastic systems, and to the non-linear stochastic optimal control of partially observable linear systems [14].

In the previous study of non-linear stochastic optimal control, the random excitation is assumed to be Gaussian white noise or wideband random process. However, quite often, the random loading of structures is a narrowband random process. One example is the excitation experienced by the cables in cable-stayed bridges caused by the deck and/or towers in vortex shedding and buffeting. The popular model of a narrowband random excitation is the response of a second order linear filter to Gaussian white noise. A rather new model of narrowband random excitation is the so-called bounded noise. A bounded noise is a harmonic function with constant amplitude and stochastic frequency and phase. This model was first proposed by Stratonovich [15] and has been used by many researchers in the study of stochastic stability of linear systems [16–18] and chaotic motion of Duffing oscillator [19].

In the present paper, the non-linear stochastic optimal control of strongly non-linear systems of single degree of freedom (s.d.o.f.) under external and/or parametric excitations of bounded noise is investigated. For this purpose, a stochastic averaging procedure for strongly non-linear systems under external and/or parametric excitations of bounded noise without control is first developed. The procedure is then applied to the controlled systems to obtain the controlled averaged Itô stochastic differential equations, from which the dynamical programming equations for finite and semi-infinite time interval controls are derived based on the stochastic dynamical programming principle. The optimal control law is determined from solving the dynamical programming equations. The Fokker–Planck–Kolmogorov (FPK) equation associated with fully completed averaged Itô equations is solved to yield the response of optimally controlled systems. The feedback stabilization of the systems is also studied and the stability of the controlled systems is determined by evaluating its Lyapunov exponent. The application and effectiveness of the control strategy are illustrated with the control of cable vibration in cable-stayed bridges.

2. The stochastic averaging method

The stochastic averaging method for s.d.o.f. strongly non-linear systems subject to external and/or parametric excitations of Gaussian white noises, wideband random processes and combined Gaussian white noise and harmonic functions, respectively, has been developed [20–22]. Here, we develop the stochastic averaging method for strongly non-linear systems under external and/or parametric excitations of narrowband random process represented by bounded noise. The equation of motion of the system studied is of the form

$$\ddot{X} + g(x) = \varepsilon f(X, \dot{X}) + \varepsilon h(X, \dot{X})\xi(t), \tag{1}$$

where g represents a strongly non-linear restoring force; ε is a small parameter; εf represents light linear and/or non-linear damping forces; εh denotes the amplitude of excitation; $\xi(t)$ is bounded noise of the form

$$\xi(t) = \sin(\Omega t + \sigma \bar{B}(t) + \Delta), \tag{2}$$

in which Ω and σ are constants, Ω is the center frequency, σ represents the intensity of frequency stochastic perturbation; $\bar{B}(t)$ is the standard Wiener process; Δ is a random phase uniformly distributed in $[0, 2\pi]$. It can be shown that $\xi(t)$ is a stationary process in wide sense with spectral density

$$S(\omega) = \frac{\sigma^2}{4\pi} \frac{\omega^2 + \Omega^2 + \sigma^4/4}{[(\omega^2 - \Omega^2 - \sigma^4/4)^2 + \sigma^4\omega^2]} \tag{3}$$

and auto-correlation function [23]

$$R(\tau) = \frac{1}{2}\exp\left(-\frac{\sigma^2}{2}|\tau|\right)\cos\Omega\tau. \tag{4}$$

The bandwidth of $\xi(t)$ depends mainly on σ. It is narrow band when σ is small and wide band when σ is large.

Suppose that the non-linear conservative oscillator

$$\ddot{x} + g(x) = 0 \tag{5}$$

possesses a family of periodic solutions in phase plane $(x, \dot{x})$ surrounding $(d^*, 0)$. The periodic solution can be expressed as

$$x(t) = a^* \cos\varphi^*(t) + d^*, \tag{6}$$

$$\dot{x}(t) = -a^* v(a^*, \varphi^*) \sin\varphi^*(t), \tag{7}$$

where

$$\varphi^*(t) = \tau(t) + \theta^*, \tag{8}$$

$$v(a^*, \varphi^*) = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \left\{\frac{2[U(a^* + d^*) - U(a^*\cos\varphi^* + d^*)]}{a^{*2}\sin^2\varphi^*}\right\}^{1/2}. \tag{9}$$

a^* and d^* are constants and related by the potential energy

$$U(x) = \int_0^x g(u)\,\mathrm{d}u \tag{10}$$

and the total energy

$$H = \dot{x}^2/2 + U(x) \tag{11}$$

as follows:

$$U(a^* + d^*) = U(d^* - a^*) = H. \tag{12}$$

$\cos\varphi^*(t)$ and $\sin\varphi^*(t)$ are the called generalized harmonic functions. Obviously, a^*, $v(a^*, \varphi^*)$ and θ^* are the amplitude, instantaneous frequency and phase, respectively, of the oscillator.

Expanding v^{-1} into Fourier series

$$v^{-1}(a^*, \varphi^*) = C_0(a^*) + \sum_{n=1}^{\infty} C_n(a^*)\cos n\varphi^* \tag{13}$$

and then integrating Eqs. (9) and (13) with respect to τ yield

$$t = C_0(a^*)\tau + \sum_{n=1}^{\infty} \frac{1}{n} C_n(a^*)\sin n\varphi^*. \tag{14}$$

Letting $\tau = 2\pi$ in Eq. (14) leads to the average period

$$T(a^*) = 2\pi C_0(a^*) \tag{15}$$

and average frequency

$$\omega(a^*) = 1/C_0(a^*) \tag{16}$$

of the oscillator.

Now consider the response of system (1). Since ε is small, it can be assumed that the solution of the system is of the form

$$X(t) = A^* \cos \Phi^*(t) + D^*, \tag{17}$$

$$\dot{X}(t) = -A^* v(A^*, \Phi^*)\sin \Phi^*(t), \tag{18}$$

where

$$\Phi^*(t) = \tau(t) + \Theta^*(t), \tag{19}$$

$$v(A^*, \Phi^*) = \frac{\mathrm{d}\tau}{\mathrm{d}t} = \left\{ \frac{2[U(A^* + D^*) - U(A^* \cos \Phi^* + D^*)]}{A^{*2} \sin^2 \Phi^*} \right\}^{1/2}, \tag{20}$$

where $A^*, D^*, \Phi^*, \Theta^*, \tau$ and v are random processes. Differentiating Eq. (17) with respect to t and equating the resultant to Eq. (18) yield

$$\dot{A}^*(\cos \Phi^* + \bar{h}) - \dot{\Theta}^* A^* \sin \Phi^* = 0, \tag{21}$$

where

$$\bar{h} = \frac{\mathrm{d}D^*}{\mathrm{d}A^*} = \frac{g(-A^* + D^*) + g(A^* + D^*)}{g(-A^* + D^*) - g(A^* + D^*)} \tag{22}$$

which is obtained from differentiation of Eq. (12) with respect to a^*. Differentiating Eq. (18) with respect to t and then substituting the resultant into Eq. (1) lead to

$$\begin{aligned} &\dot{A}^* \left\{ v(A^*, \Phi^*)\sin \Phi^* + A^* \frac{\partial}{\partial A^*}[v(A^*, \Phi^*)\sin \Phi^*] \right\} + \dot{\Theta}^* \frac{\partial}{\partial \Phi^*}[Av(A^*, \Phi^*)\sin \Phi^*] \\ &\quad = -\varepsilon f(A^* \cos \Phi^* + D^*, -A^* v(A^*, \Phi^*)\sin \Phi^*) \\ &\qquad - \varepsilon h(A^* \cos \Phi^* + D^*, -A^* v(A^*, \Phi^*)\sin \Phi^*)\xi(t). \end{aligned} \tag{23}$$

Solving Eqs. (21) and (23) for $\dot{A}^*$ and $\dot{\Theta}^*$, we obtain

$$\begin{aligned} \frac{\mathrm{d}A^*}{\mathrm{d}t} &= \varepsilon F_1(A^*, \Phi^*, \Omega t + \Lambda^*), \\ \frac{\mathrm{d}\Theta^*}{\mathrm{d}t} &= \varepsilon F_2(A^*, \Phi^*, \Omega t + \Lambda^*), \end{aligned} \tag{24}$$

where $\Lambda^* = \sigma B(t) + \Delta$,

$$\begin{aligned} \varepsilon F_1 = &- \frac{\varepsilon A^*}{g(A^* + D^*)(1 + \bar{h})}[f(A^* \cos \Phi^* + D^*, -A^* v(A^*, \Phi^*)\sin \Phi^*) \\ &+ h(A^* \cos \Phi^* + D^*, \; -A^* v(A^*, \Phi^*)\sin \Phi^*)\sin(\Omega t + \Lambda^*)]v(A^*, \Phi^*)\sin \Phi^*, \end{aligned}$$

$$\begin{aligned}\varepsilon F_2 = &- \frac{\varepsilon}{g(A^* + D^*)(1+\bar{h})}[f(A^* \cos \Phi^* + D^*, -A^* v(A^*, \Phi^*)\sin \Phi^*) \\ &+ h(A^* \cos \Phi^* + D^*, \; -A^* v(A^*, \Phi^*)\sin \Phi^*)\sin(\Omega t + \Lambda^*)]v(A^*, \Phi^*)(\cos \Phi^* + \bar{h}).\end{aligned} \tag{25}$$

Since we are interested in narrowband excitation and resonant case, it is assumed that σ is small and

$$\frac{\Omega}{\omega(A^*)} = \frac{q}{p} + \delta, \tag{26}$$

where p and q are relatively prime positive small integers and δ is a detuning parameter of the order of ε. Using Eqs. (14) and (26), we obtain

$$\Omega t = \frac{q}{p}\Phi^* + \delta\tau - \frac{q}{p}\Theta^* + \Omega \sum_{n=1}^{\infty} \frac{1}{n} C_n(A^*)\sin n\Phi^*. \tag{27}$$

Introduce new variable

$$\Gamma^* = \delta\tau - \frac{q}{p}\Theta^* + \Lambda^*, \tag{28}$$

which denotes the phase difference of the excitation and response. Then

$$\Omega t + \Lambda^* = \Psi + \Gamma^*, \tag{29}$$

where

$$\Psi = \Psi(A^*, \Phi^*) = \frac{q}{p}\Phi^* + \Omega \sum_{n=1}^{\infty} \frac{1}{n} C_n(A^*)\sin n\Phi^*. \tag{30}$$

Regarding Eq. (28) as a transformation from Θ^* to Γ^*, we obtain the following Itô equations from Eq. (24):

$$\begin{aligned}\mathrm{d}A^* &= \varepsilon F_1(A^*, \Phi^*, \Psi + \Gamma^*)\mathrm{d}t, \\ \mathrm{d}\Gamma^* &= \left[\left(\frac{\Omega}{\omega(A^*)} - \frac{q}{p}\right)v(A^*, \Phi^*) - \frac{q}{p}\varepsilon F_2(A^*, \Phi^*, \Psi + \Gamma^*)\right]\mathrm{d}t + \sigma\,\mathrm{d}B(t).\end{aligned} \tag{31}$$

Note that ε and δ are small so that A^* and Γ^* are slowly varying processes while Φ^* is rapidly varying process. Averaging the right hand side of Eq. (31) with respect to Φ^* leads to the following averaged Itô equations:

$$\begin{aligned}\mathrm{d}A &= m_1(A, \Gamma)\,\mathrm{d}t, \\ \mathrm{d}\Gamma &= m_2(A, \Gamma)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t),\end{aligned} \tag{32}$$

where

$$\begin{aligned}m_1 &= \langle \varepsilon F_1(A^*, \Phi^*, \Psi + \Gamma^*)\rangle_{\Phi^*}, \\ m_2 &= \left\langle \left[\left(\frac{\Omega}{\omega(A^*)} - \frac{q}{p}\right)v(A^*, \Phi^*) - \frac{q}{p}\varepsilon F_2(A^*, \Phi^*, \Psi + \Gamma^*)\right]\right\rangle_{\Phi^*}.\end{aligned} \tag{33}$$

and $\langle \cdot \rangle_{\Phi^*}$ denotes the averaging operation. Averaging with respect to Φ^* removes the rapid fluctuation of small amplitude superposed on the slow drift and smoothens the response.

The FPK equation associated with Itô averaged equation (32) is

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial a}(m_1 p) - \frac{\partial}{\partial \gamma}(m_2 p) + \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial \gamma^2}, \tag{34}$$

where $p = p(a, \gamma, t | a_0, \gamma_0)$ or $p = p(a, \gamma, t)$ depending on the form of initial condition given, i.e.,

$$p = \delta(a - a_0)\delta(\gamma - \gamma_0), \quad t = 0 \tag{35}$$

or

$$p = p(a_0, \gamma_0), \quad t = 0. \tag{36}$$

The boundary conditions with respect to a are

$$p = \text{finite}, \quad a = 0 \tag{37}$$

and

$$p, \partial p / \partial a \to 0 \quad \text{as } a \to \infty. \tag{38}$$

The conditions with respect to γ is periodic, i.e.,

$$p|_{\gamma} = p|_{\gamma + 2n\pi}, \tag{39}$$

$$(\partial p / \partial \gamma)|_{\gamma} = (\partial p / \partial \gamma)|_{\gamma + 2n\pi}. \tag{40}$$

Boundary condition (37) is qualitative and can be made to be quantitative one by using Eq. (34) and m_1, m_2 at $a = 0$. FPK equation (34) together with its initial and boundary conditions can be solved numerically by using the method of path integrations as did in Ref. [22]. The statistics of the response of system (1) can be obtained from the solution to FPK equation (34).

3. Non-linear stochastic optimal control

Now consider the non-linear stochastic optimal control of system (1). The equation of controlled system is of the form

$$\ddot{X} + g(X) = \varepsilon f(X, \dot{X}) + \varepsilon h(X, \dot{X})\xi(t) + \varepsilon e_k(X, \dot{X})u_k, \quad k = 1, 2, \ldots, m, \tag{41}$$

where $u_k = u_k(X, \dot{X})$ are feedback control, εe_k are their amplitudes and the other notations are the same as in Eq. (1). By applying the stochastic averaging method developed in the last section to system (41), the following averaged Itô equations are obtained:

$$\begin{aligned} \mathrm{d}A &= m_1^u(A, \Gamma, \langle \mathbf{u} \rangle)\,\mathrm{d}t, \\ \mathrm{d}\Gamma &= m_2^u(A, \Gamma, \langle \mathbf{u} \rangle)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \end{aligned} \tag{42}$$

where $\mathbf{u} = [u_1, u_2, \ldots, u_m]^{\mathrm{T}}$

$$m_1^u = m_1(A, \Gamma) - \left\langle \frac{\varepsilon A^* u_k}{g(A^* + D^*)(1 + \bar{h})} e_k(A^* \cos\Phi^* + D^*, -A^* v(A, \Phi^*)\sin\Phi^*) v(A^*, \Phi^*) \sin\Phi^* \right\rangle_{\Phi^*},$$

$$
\begin{aligned}
m_2^u = {} & m_2(A, \Gamma) \\
& + \left\langle \frac{q}{p} \frac{\varepsilon u_k}{g(A^* + D^*)(1+\bar{h})} e_k(A^* \cos \Phi^* + D^*, -A^* v(A^*, \Phi^*) \sin \Phi^*) v(A^*, \Phi^*) \right. \\
& \left. \times (\cos \Phi^* + \bar{h}) \right\rangle_{\Phi^*}.
\end{aligned} \tag{43}
$$

m_1 and m_2 are defined by Eq. (33). Eq. (42) implies that (A, Γ) is a controlled diffusion process, to which the stochastic dynamical programming principle [8–10] can be applied.

The optimal feedback control law depends on the objective of the control, which is expressed in terms of a performance index. In the present section, we are interested in the control of the response in a finite or semi-infinite time interval. For this purpose, the following form of a performance index is taken:

$$
J = E\left[\int_0^{t_f} L(A, \Gamma, \langle \mathbf{u} \rangle)\, \mathrm{d}t + G(A(t_f), \Gamma(t_f)) \right] \tag{44}
$$

for finite time interval control, or

$$
J = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} L(a, \gamma, \langle \mathbf{u} \rangle)\, \mathrm{d}t \tag{45}
$$

for semi-infinite time interval control. In Eqs. (44) and (45), L is called cost function and G is final cost. The control with performance (45) is called ergodic control. Based on the stochastic dynamical programming principle [8–10], the following dynamical programming equation can be established:

$$
\frac{\partial V}{\partial t} = -\min_u \left\{ L(a, \gamma, \langle \mathbf{u} \rangle) + m_1^u \frac{\partial V}{\partial a} + m_2^u \frac{\partial V}{\partial \gamma} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial \gamma^2} \right\} \tag{46}
$$

for finite time interval control, or

$$
\eta = \min_u \left\{ L(a, \gamma, \langle \mathbf{u} \rangle) + m_1^u \frac{\partial V}{\partial a} + m_2^u \frac{\partial V}{\partial \gamma} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial \gamma^2} \right\} \tag{47}
$$

for semi-infinite time interval control. In Eq. (46)

$$
V = V(a, \gamma, t) = E\left[G(A(t_f), \Gamma(t_f)) - \int_{t_f}^{t} L(A, \Gamma, \langle \mathbf{u} \rangle)\, \mathrm{d}t \right] \tag{48}
$$

is called value function with final time condition

$$
V(a, \gamma, t_f) = E[G(A(t_f), \Gamma(t_f)]. \tag{49}
$$

In Eq. (47),

$$
\eta = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} L(a, \gamma, \langle \mathbf{u}^* \rangle)\, \mathrm{d}t, \tag{50}
$$

where η is optimal average cost and $\mathbf{u}^*$ is the optimal control.

The optimal feedback control $\mathbf{u}^*$ is determined by minimizing the right hand side of Eq. (46) or (47) with respect to $\mathbf{u}$. Suppose that

$$L(a,\gamma,\langle\mathbf{u}\rangle) = L_1(a,\gamma) + \langle\mathbf{u}^T\mathbf{R}\mathbf{u}\rangle, \tag{51}$$

where $\mathbf{R} = diag(R_1, R_2, \ldots, R_m)$ with $R_k > 0$. Then the optimal feedback control law is

$$u_k^* = \frac{1}{2R_k}\frac{\varepsilon e_k(a^*,\varphi^*)v(a^*,\varphi^*)}{g(a^*+d^*)(1+\bar{h})}\left[a^*\sin\varphi^*\frac{\partial V}{\partial a^*} - \frac{q}{p}(\cos\varphi^*+\bar{h})\frac{\partial V}{\partial\gamma^*}\right],\quad k=1,2,\ldots,m. \tag{52}$$

Substituting u_k^* into Eq. (43) for u_k and averaging the second terms on the right hand side of Eq. (43) with respect to Φ^* yield

$$\begin{aligned} m_1^c(A,\Gamma,V) = m_1(A,\Gamma) - \Bigg\langle & \frac{\varepsilon A^* u_k^*}{g(A^*+D^*)(1+\bar{h})} e_k(A^*\cos\Phi^* + D^*, \\ & - A^* v(A^*,\Phi^*)\sin\Phi^*)v(A^*,\Phi^*)\sin\Phi^*\Bigg\rangle_{\Phi^*}, \end{aligned} \tag{53}$$

$$\begin{aligned} m_2^c(A,\Gamma,V) = m_2(A,\Gamma) + \frac{q}{p}\Bigg\langle & \frac{\varepsilon u_k^*}{g(A^*+D^*)(1+\bar{h})} e_k(A^*\cos\Phi^* + D^*, \\ & - A^* v(A^*,\Phi^*)\sin\Phi^*)v(A^*,\Phi^*)(\cos\Phi^*+\bar{h}))\Bigg\rangle_{\Phi^*}. \end{aligned}$$

The final dynamical programming equation is obtained from Eq. (46) or (47) by replacing m_1^u, m_2^u with m_1^c, m_2^c, respectively. i.e.,

$$-\frac{\partial V}{\partial t} = L(a,\gamma,\langle\mathbf{u}^*\rangle) + m_1^c\frac{\partial V}{\partial a} + m_2^c\frac{\partial V}{\partial\gamma} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial\gamma^2} \tag{54}$$

for finite time interval control, or

$$\eta = L(a,\gamma,\langle\mathbf{u}^*\rangle) + m_1^c\frac{\partial V}{\partial a} + m_2^c\frac{\partial V}{\partial\gamma} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial\gamma^2} \tag{55}$$

for semi-infinite time interval control. The averaged Itô equations for controlled system (41) are obtained from Eq. (42) by replacing m_1^u, m_2^u with m_1^c, m_2^c, respectively, i.e.,

$$\begin{aligned} \mathrm{d}A &= m_1^c(A,\Gamma,V)\,\mathrm{d}t, \\ \mathrm{d}\Gamma &= m_2^c(A,\Gamma,V)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t). \end{aligned} \tag{56}$$

Solving Eq. (54) or (55) for $V = V(a,\gamma,t)$ or $V = V(a,\gamma)$ and then substituting it into Eq. (56) yield the fully completed averaged Itô equations for A, Γ. The response statistics of the optimally controlled system (41) can be obtained from solving the FPK equation associated with Itô equation (56) by using the method of path integration.

4. Vibration control of cables under external excitation

The bridge stay cables are susceptible to vibration under wind loading due to their large flexibility, relative small mass and very low inherent damping. The active control of cable vibration has been studied by numerous researchers [24–28]. In almost all these studies, the wind loading is modeled as harmonic function. However, wind loading is actually random. As indicated by Lin et al. [17] that the bounded noise is a good model for wind turbulence. So, here we take it as the model of cable excitation. Since we are interested in the control of resonant vibration of cables and usually only one mode of a cable is resonant, here the first mode of the cable is taken as an example. For the control of the other mode of the cable, only the modal parameters needed to be changed. The equation of the controlled system is of the form [29]

$$\ddot{X} + 2\zeta\omega_1\dot{X} + \omega_1^2 X + \beta X^3 = \mu\xi(t) + u_1(t) + Xu_2(t), \tag{57}$$

where $\omega_1^2 = \pi^2 T_0/L^2\rho$ is the natural frequency of first mode of cable; T_0 is the tension of the cable without vibration; L is the initial length of the cable; ρ is the mass per unit length of cable; ζ is the damping ratio of the first mode of the cable; constant $\beta = \pi^2\omega_1^2/4LZ_0$ represents the intensity of non-linearity of the cable in large amplitude vibration; Z_0 is the initial extension due to tension T_0; $\mu\xi(t)$ is the first modal excitation; $u_1(t) = [2\sin(\pi s/L)/\rho L]F(t)$, $F(t)$ is the transverse control force and s is the distance along the cable axis from acting point of control force $F(t)$ to the end of the cable; u_2 is the active stiffness control via boundary motion [24], see Fig. 1.

For the present example,

$$\begin{aligned} &g(x) = \omega_1^2 x + \beta x^3, \\ &U(x) = \omega_1^2 x^2/2 + \beta x^4/4, \\ &d^* = 0,\ \bar{h} = 0, \\ &v(a^*, \varphi^*) = (\omega_1^2 + 3\beta a^{*^2}/4 + \beta a^{*^2}\cos 2\varphi^*/4)^{1/2}. \end{aligned} \tag{58}$$

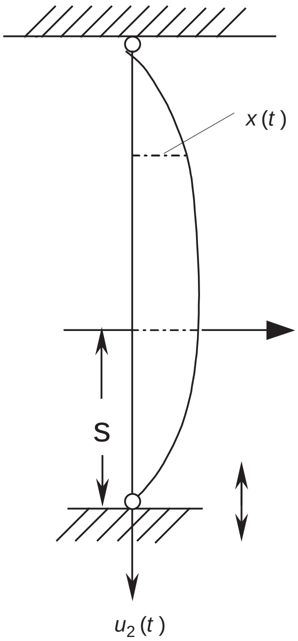

Fig. 1. A cable under control.

We are interested in the case of primary resonance, i.e.,

$$p = q = 1, \quad \Omega/\omega(a) = 1 + \delta. \tag{59}$$

In this case,

$$m_1^u = m_1(A, \Gamma) - \frac{v(A^*, \Phi^*)\sin\Phi^*}{\omega_1^2 + \beta A^{*2}}[u_1 + (A^*\cos\Phi^*)u_2]$$

$$m_2^u = m_2(A, \Gamma) + \frac{v(A^*, \Phi^*)\cos\Phi^*}{(\omega_1^2 + \beta A^{*2})A^*}[u_1 + (A^*\cos\Phi^*)u_2],$$

$$m_1(A, \Gamma) = -\zeta\omega_1 A\frac{\omega_1^2 + 5\beta A^2/8}{\omega_1^2 + \beta A^2} - \frac{\mu\cos\Gamma}{\omega_1^2 + \beta A^2} \times \left\langle v(A^*, \Phi^*)\sin(\Phi^* + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(A^*)\sin n\Phi^*)\sin\Phi^* \right\rangle_{\Phi^*},$$

$$m_2(A, \Gamma) = \left(\frac{\Omega}{\omega(A)} - 1\right)\langle v(A^*, \Phi^*)\rangle_{\Phi^*} + \frac{\mu\sin\Gamma}{(\omega_1^2 + \beta A^2)A} \times \left\langle v(A^*, \Phi^*)\cos(\Phi^* + \Omega\sum_{n=1}^{\infty}\frac{1}{n}C_n(A^*)\sin n\Phi^*)\cos\Phi^* \right\rangle_{\Phi^*}. \tag{60}$$

Consider the ergodic control with cost function of the form of Eq. (51) with $m = 2$. According to Eq. (52), the optimal control law is

$$u_1^* = \frac{1}{2R_1}\frac{v(a^*, \varphi^*)}{(\omega_1^2 + \beta a^{*2})a^*}\left(a^*\sin\varphi^*\frac{\partial V}{\partial a^*} - \cos\varphi^*\frac{\partial V}{\partial\gamma^*}\right),$$

$$u_2^* = \frac{1}{2R_2}\frac{v(a^*, \varphi^*)\cos\varphi^*}{\omega_1^2 + \beta a^{*2}}\left(a^*\sin\varphi^*\frac{\partial V}{\partial a^*} - \cos\varphi^*\frac{\partial V}{\partial\gamma^*}\right). \tag{61}$$

The final dynamical programming equation is of the form

$$\eta = L_1(a, \gamma) + m_1(a, \gamma)\frac{\partial V}{\partial a} + m_2(a, \gamma)\frac{\partial V}{\partial\gamma} - \left(\frac{m_{11}(a)}{4R_1} + \frac{m_{21}(a)}{4R_2}\right)\left(\frac{\partial V}{\partial a}\right)^2 - \left(\frac{m_{12}(a)}{4R_1} + \frac{m_{22}(a)}{4R_2}\right)\left(\frac{\partial V}{\partial\gamma}\right)^2 + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial\gamma^2}, \tag{62}$$

where

$$m_{11}(a) = \frac{\omega_1^2 + 5\beta a^2/8}{2(\omega_1^2 + \beta a^2)^2}, \quad m_{12}(a) = \frac{\omega_1^2 + 7\beta a^2/8}{2a^2(\omega_1^2 + \beta a^2)},$$

$$m_{21}(a) = \frac{(\omega_1^2 + 3\beta a^2/4)a^2}{8(\omega_1^2 + \beta a^2)^2}, \quad m_{22}(a) = \frac{3(\omega_1^2 + 11\beta a^2/12)}{8(\omega_1^2 + \beta a^2)^2}. \tag{63}$$

Eq. (62) can be solved by expanding V and the coefficients of the equation into Fourier series with respect to γ and then expanding all the Fourier coefficients into Taylor series. See the Appendix for detail. It is shown that an approximate solution of Eq. (62) with enough accuracy is of the form

$$V(a,\gamma) = V_{02}a^2 + V_{11}^c a\cos\gamma + V_{11}^s a\sin\gamma \tag{64}$$

where V_{02}, V_{11}^c and V_{11}^s are constants depending on the system parameters. In this case, the fully completed averaged Itô equations of the optimally controlled system are

$$\begin{aligned}
\mathrm{d}A &= [m_1(A,\Gamma) + m_1^*(A,\Gamma)]\,\mathrm{d}t,\\
\mathrm{d}\Gamma &= [m_2(A,\Gamma) + m_2^*(A,\Gamma)]\,\mathrm{d}t + \sigma\,\mathrm{d}B(t),
\end{aligned} \tag{65}$$

where m_1 and m_2 are defined by Eq. (60),

$$\begin{aligned}
m_1^* &= -\frac{\omega_1^2 + 5\beta A^2/8}{4R_1(\omega_1^2+\beta A^2)^2}(2V_{02}A + V_{11}^c\cos\Gamma + V_{11}^s\sin\Gamma),\\
m_2^* &= \frac{\omega_1^2 + 7\beta A^2/8}{4R_1 A(\omega_1^2+\beta A^2)^2}(V_{11}^c\sin\Gamma - V_{11}^s\cos\Gamma).
\end{aligned} \tag{66}$$

Solving the FPK equation associated with Itô equations (65) by using the method of path integration yields stationary probability density $p(a,\gamma)$. A numerical result is shown in Fig. 2(a). To check the result, the associated simulation result is shown in Fig. 2(b). The corresponding results for the system without control obtained by using the proposed stochastic averaging method and from digital simulation are shown in Figs. 3(a), and (b), respectively. It is seen that the probability density is bimodal and jump may occur in the response of uncontrolled system while probability density is unimodal and no jump may occur in the response of controlled system. The mean and mean square of vibration amplitude are reduced by control from 1.2082 to 0.4678, and from 1.6311 to 0.2297, respectively. One more set of results are shown in Figs. 4 and 5, where the mean and mean square of vibration amplitude are also reduced remarkably (from 0.8618 to 0.4978, and from 0.7609 to 0.2393, respectively) by control. It is seen that the difference in peak locations in a direction of the probability densities obtained by using the analytical method and from digital simulation is very small, which implies that the difference in mean responses predicted by using the two methods is small. The difference in peak heights of the two probability densities is slightly larger, which implies that the difference in the mean square response predicted by using the two methods is slightly larger.

To compare the proposed control strategy with LQR controller designed for the linearized system model, consider the degenerated linear equation of (57) with $u_2 = 0$, i.e.,

$$\ddot{X} + 2\zeta\omega_1\dot{X} + \omega_1^2 X = \mu\xi(t) + u_1(t). \tag{67}$$

For this system, the optimal control law (61) is reduced to

$$u_1^* = \frac{1}{2R_1\omega_1 a^*}\left(a^*\sin\varphi^*\frac{\partial V}{\partial a^*} - \cos\varphi^*\frac{\partial V}{\partial\gamma^*}\right) \tag{68}$$

and the solution to the final dynamical programming equation (62) is

$$V = V_{02}a^2 + V_{11}^c a\cos\gamma + V_{11}^s a\sin\gamma. \tag{69}$$

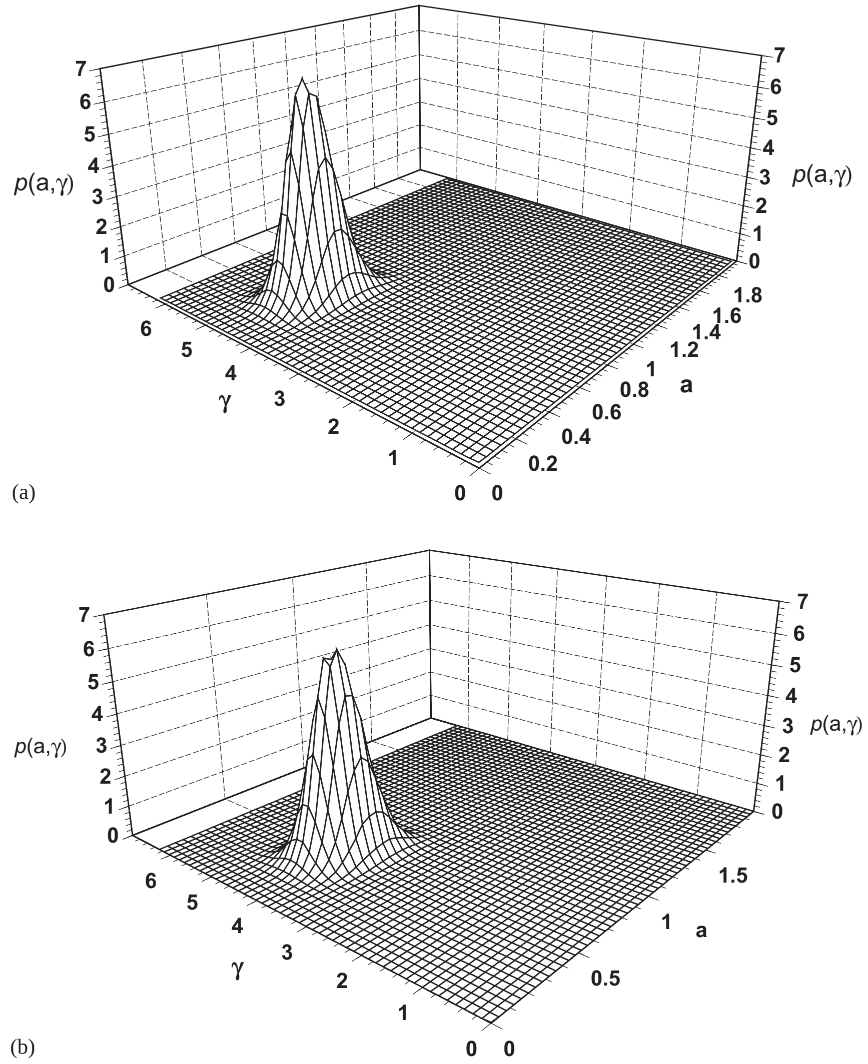

Fig. 2. Stationary probability density $p(a,\gamma)$ of controlled system (57). $\omega_1 = 1.0$, $\zeta = 0.05$, $\alpha = 0.3$, $\Omega = 1.2$, $\mu = 0.2$, $\sigma^2 = 0.01$, $L_{102} = 0.998$, $L_{104} = 0$, $L_{11}^c = L_{11}^s = 0$, $\eta = 1.0$, $R_1 = 1$, $R_2 = 10^5$; (a) analytical result; (b) from digital simulation.

Substituting Eq. (69) into Eq. (68), one obtains

$$u_1^* = -\frac{V_{02}}{R_1\omega_1^2}\dot{x} + \frac{V_{11}^c}{2R_1\omega_1}\sin(\Omega t + \sigma B(t) + \Delta) - \frac{V_{11}^s}{2R_1\omega_1}\cos(\Omega t + \sigma \bar{B}(t) + \Delta) \tag{70}$$

which depends on $(\eta - L_{100})/R_1$, L_{111}^c/R_1 and L_{111}^s/R_1.

To use the LQR controller, let $X = [X_1, X_2]^{\mathrm{T}}$, where $X_1 = X$, $X_2 = \dot{X}$ and rewrite Eq. (67) in the form of state equation

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{B}[u(t) + \mu\xi(t)], \tag{71}$$

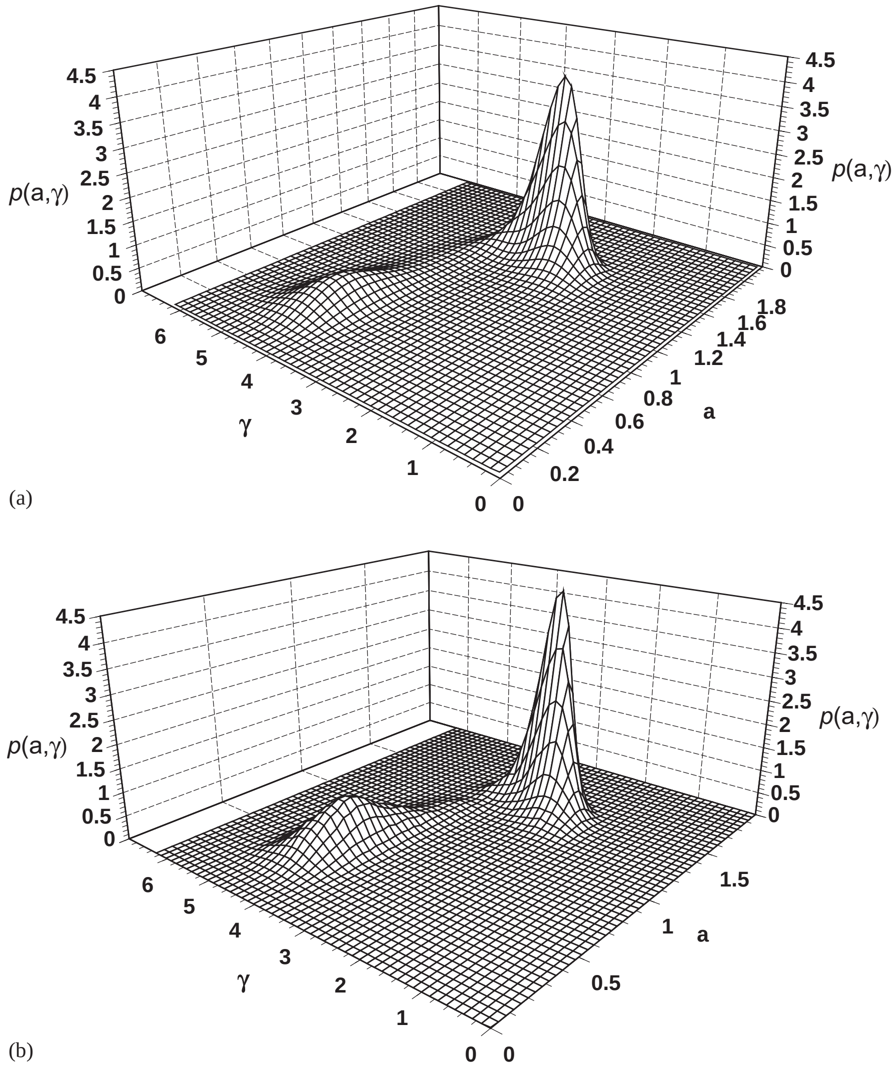

Fig. 3. Stationary probability density $p(a,\gamma)$ of system (57) without control ($u_1 = u_2 = 0$). The parameters are the same as those in Fig. 2; (a) analytical result; (b) from digital simulation.

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_1^2 & -2\zeta\omega_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{72}$$

For semi-infinite time interval control, the performance index for LQR is of the form

$$J = \lim_{t_f \to \infty} \frac{1}{t_f} \int_0^{t_f} (\mathbf{X}^{\mathrm{T}}\mathbf{S}\mathbf{X} + Ru^2)\,\mathrm{d}t, \tag{73}$$

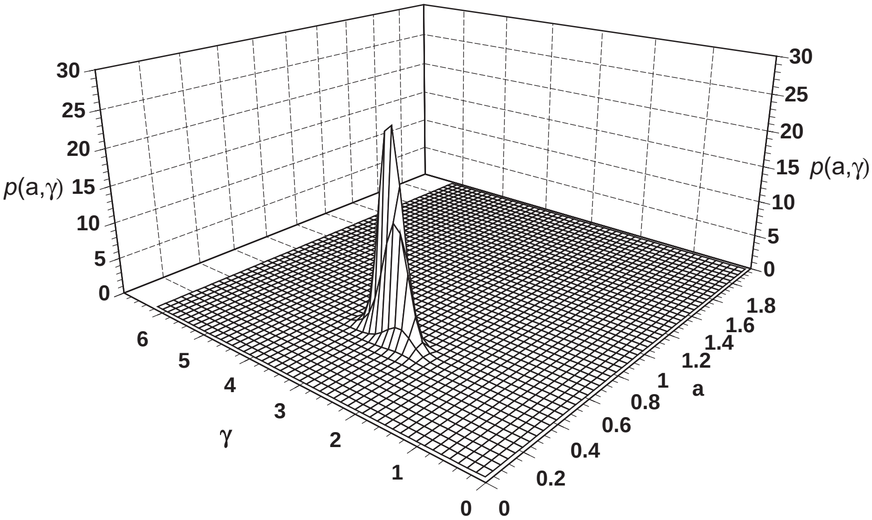

Fig. 4. Stationary probability density $p(a,\gamma)$ of controlled system (57). $\omega_1 = 1.0$, $\zeta = 0.05$, $\alpha = 0.3$, $\Omega = 0.2$, $\mu = 0.2$, $\sigma^2 = 0.01$, $L_{102} = 0.985$, $L_{104} = 0$, $L^c_{11} = L^s_{11} = 0$, $\eta = 1.0$, $R_1 = 1.0$, $R_2 = 10^5$.

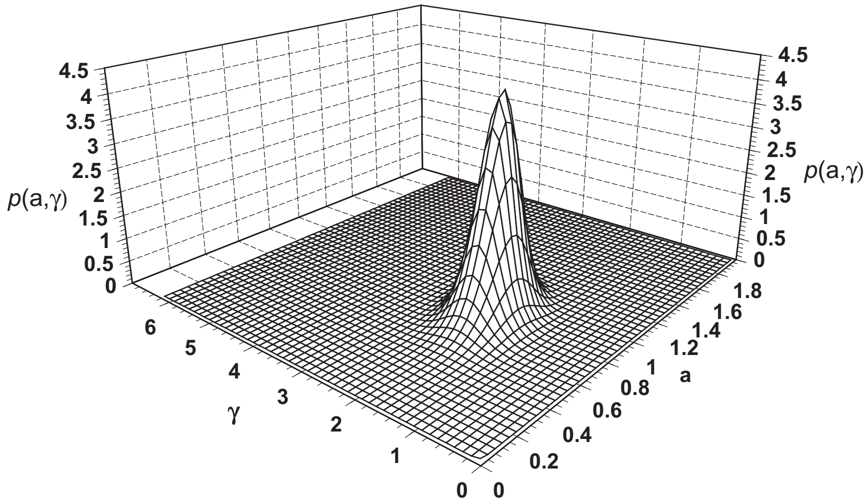

Fig. 5. Stationary probability density $p(a,\gamma)$ of system (57) without control ($u_1 = u_2 = 0$). The other parameters are the same as those in Fig. 4.

where R is a positive constant and

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \tag{74}$$

is a non-negative matrix. The optimal control force is of the form

$$u^* = -\frac{1}{R}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{X} = -\frac{p_{12}}{R}X - \frac{p_{22}}{R}\dot{X}, \tag{75}$$

where matrix $\boldsymbol{P}$ satisfies the following Ricatti equation:

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{S} - R\mathbf{PBB}^{\mathrm{T}}\mathbf{P} = 0. \tag{76}$$

u^* in Eq. (75) depends on S_{11}/R and S_{22}/R (in case $S_{12} = S_{21} = 0$). Substituting Eq. (68) or (75) into Eq. (60) and then into Eq. (65) after averaging lead to the fully completed averaged Itô equation. The response of optimally controlled cable is obtained from solving the FPK equation associated with the completely averaged Itô equation.

The following two criteria are introduced for comparison:

$$K_1 = \frac{\sigma_x^u - \sigma_x^c}{\sigma_x^u} \times 100\%, \tag{77}$$

$$K_2 = \frac{K_1}{(\sigma_u/\sigma_e)}, \tag{78}$$

where σ_x^u and σ_x^c stand for the standard deviations of the responses of uncontrolled and optimally controlled systems, respectively; σ_u and σ_e stand for the standard deviations of control force and excitation, respectively. K_1 denotes the percentage reduction in the standard deviation of the response, i.e., the effectiveness of a controller while K_2 the efficiency of a controller. Obviously, the larger K_1 and K_2 are, the better the control strategy is.

K_1 and K_2 versus $(\eta - L_{100})/R_1$ and L_{111}^c/R_1 obtained by using the proposed control strategy are shown in Figs. 6 and 7, respectively. K_1 and K_2 versus S_{11}/R and S_{22}/R obtained by using LQR are shown in Figs. 8 and 9, respectively. The system parameters are the same for all these figures. It is seen from these figures that the effectiveness of the proposed strategy can be slightly better than that of LQR while the efficiency of the proposed controller is much higher than that of

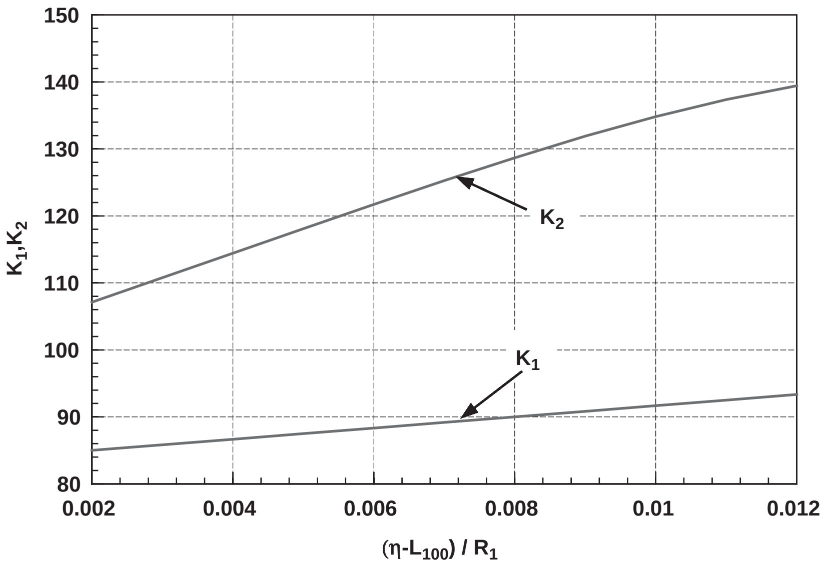

Fig. 6. K_1 and K_2 versus $(\eta - L_{100})/R_1$ for proposed control strategy. $L_{111}^c/R_1 = 0.1$, $L_{11}^s = 0.0$, $\omega_1 = \Omega = 1.0$, $\zeta = 0.05$, $\mu = 0.2$, $\sigma^2 = 0.0001$.

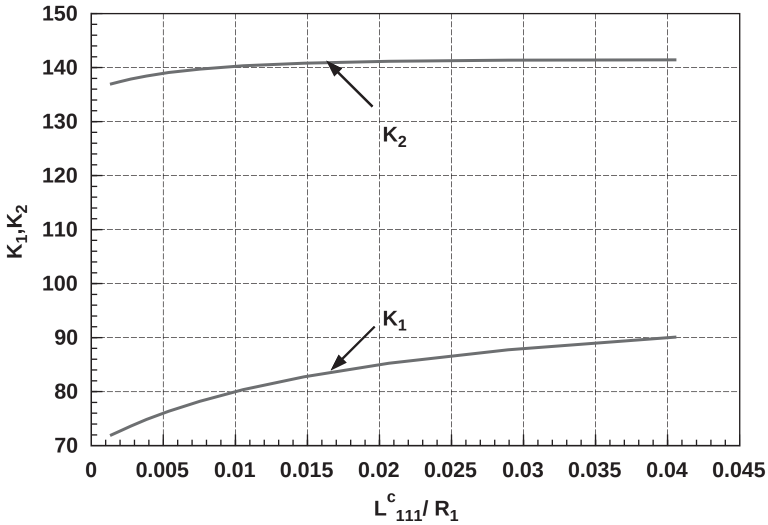

Fig. 7. K_1 and K_2 versus L^c_{111}/R_1, $(\eta - L_{100})/R_1 = 0.1$ for proposed control strategy. $L^s_{11} = 0.0$, $\omega_1 = \Omega = 1.0$, $\zeta = 0.05$, $\mu = 0.2$, $\sigma^2 = 0.0001$.

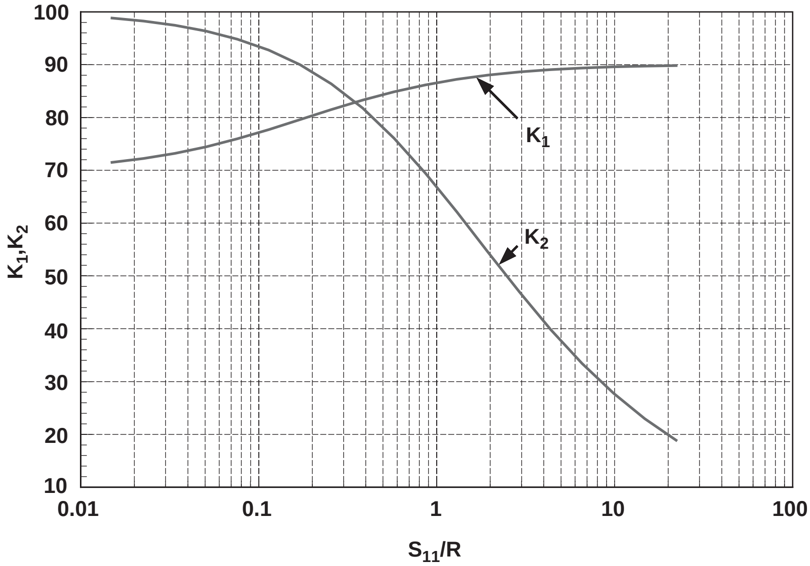

Fig. 8. K_1 and K_2 versus S_{11}/R for LQR controller. $S_{22}/R = 0.1$, $\omega_1 = \Omega = 1.0$, $\zeta = 0.05$, $\mu = 0.2$, $\sigma^2 = 0.0001$.

LQR. These differences in effectiveness and deficiency of the two controllers come from the difference in optimal control forces in Eqs. (68) and (75). The optimal control force in LQR increases the stiffness and damping while that in the proposed controller reduces the external excitation except increasing damping.

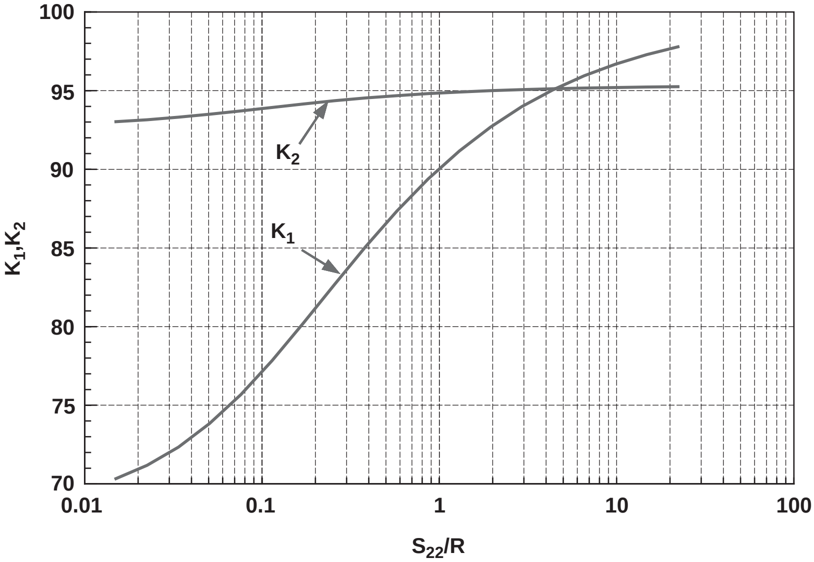

Fig. 9. K_1 and K_2 versus S_{22}/R for LQR controller. $S_{22}/R = 0.1$, $\omega_1 = \Omega = 1.0$, $\zeta = 0.05$, $\mu = 0.2$, $\sigma^2 = 0.0001$.

5. Feedback stabilization of cables under parametric excitation

The cable vibrations caused by the parametric excitation of deck and/or towers in vortex shedding and buffeting were observed in a number of cable-stayed bridges recently [30–33]. Since the deck or tower usually undergoes random vibration due to vortex shedding and buffeting, the cable is subject to randomly parametric excitation. To the authors' knowledge, the controls of the stochastic stability and parametric vibration of stay cables have not been studied. In this section, the feedback stabilization of a cable under parametric excitation is studied. Note that there will be no vibration once a cable is stable. As in the last section, only the first mode of cable is taken into account. Besides, the degenerated linear equation is used for the stabilization study since we are interested in the stability of the trivial solution of the system. The equation of the controlled system is of the form

$$\ddot{X} + 2\zeta\omega_1\dot{X} + \omega_1^2 X = \mu X\xi(t) + u_1(t), \tag{79}$$

which is the same as system (67) except the parametric excitation. Here we are interested in the case of primary parametric resonance, i.e.,

$$p = 1, \quad q = 2, \quad \Omega/\omega_1 = 2 + \delta. \tag{80}$$

The coefficients of averaged Itô equations (42) in this case are

$$m_1^u = m_1(A, \Gamma) - \left\langle \frac{\sin \Phi^*}{\omega_1} u_1 \right\rangle_{\Phi^*},$$

$$m_2^u = m_2(A, \Gamma) + \left\langle \frac{2\cos \Phi^*}{\omega_1 A^*} u_1 \right\rangle_{\Phi^*},$$

$$m_1(A,\Gamma) = -\zeta\omega_1 A - \frac{\mu A\cos\Gamma}{4\omega_1},$$
$$m_2(A,\Gamma) = (\Omega - 2\omega_1) + \frac{\mu\sin\Gamma}{2\omega_1}. \tag{81}$$

We still consider the ergodic control with cost function L of the form of Eq. (51) with $m = 1$. According to Eq. (52), the optimal control force for this system is

$$u_1^* = \frac{1}{2R_1\omega_1 a^*}\left(a^*\sin\varphi^*\frac{\partial V}{\partial a^*} - 2\cos\varphi^*\frac{\partial V}{\partial\gamma^*}\right). \tag{82}$$

The final dynamical programming equation is of the same form of Eq. (62) with m, m_2 defined by Eq. (81) and

$$m_{11}(a) = \frac{1}{2\omega_1^2},\ m_{12}(a) = \frac{2}{\omega_1^2 a^2},$$
$$m_{21}(a) = m_{22}(a) = 0. \tag{83}$$

Consider the stability of the trivial solution of system (79). The fully completed averaged Itô equations for A, Γ are of the form

$$\mathrm{d}A = m_1^c(A,\Gamma,V)\,\mathrm{d}t,$$
$$\mathrm{d}\Gamma = m_2^c(A,\Gamma,V)\,\mathrm{d}t + \sigma\,\mathrm{d}B(t), \tag{84}$$

where

$$m_1^c(A,\Gamma,V) = -\zeta\omega_1 A - \frac{\mu A}{4\omega_1}\cos\Gamma - \frac{1}{4R_1\omega_1^2}\frac{\partial V}{\partial A},$$
$$m_2^c(A,\Gamma,V) = \Omega - 2\omega_1 + \frac{\mu}{2\omega_1}\sin\Gamma - \frac{1}{R_1\omega_1^2 A^2}\frac{\partial V}{\partial\Gamma} \tag{85}$$

and V is the solution to the final dynamical programming equation

$$\eta = L_1(a,\gamma) + \left(-\zeta\omega_1 a - \frac{\mu a}{4\omega_1}\cos\gamma\right)\frac{\partial V}{\partial a} - \frac{1}{8R_1\omega_1^2}\left(\frac{\partial V}{\partial a}\right)^2$$
$$+ \left(\Omega - 2\omega_1 + \frac{\mu}{2\omega_1}\sin\gamma\right)\frac{\partial V}{\partial\gamma} - \frac{1}{2R_1\omega_1^2 a^2}\left(\frac{\partial V}{\partial\gamma}\right)^2 + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial\gamma^2}. \tag{86}$$

For the uncontrolled system, the averaged Itô equations for A and Γ are

$$\mathrm{d}A = \left(-\zeta\omega_1 A - \frac{\mu A}{4\omega_1}\cos\Gamma\right)\mathrm{d}t,$$
$$\mathrm{d}\Gamma = \left(\Omega - 2\omega_1 + \frac{\mu}{2\omega_1}\sin\Gamma\right)\mathrm{d}t + \sigma\,\mathrm{d}B(t). \tag{87}$$

Introduce new variable

$$\rho = \ln A. \tag{88}$$

The Itô equation for ρ is obtained from Eq. (87) by using Itô differential rule

$$\mathrm{d}\rho = \left(-\zeta\omega_1 - \frac{\mu}{4\omega_1}\cos\Gamma\right)\mathrm{d}t. \tag{89}$$

The Lyapunov exponent of the linear equation (79) is defined as

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\ln(X^2(t) + \dot{X}^2(t)/\omega_1^2)^{1/2} = \lim_{t\to\infty}\frac{1}{t}\ln A = \lim_{t\to\infty}\frac{1}{t}\rho(t). \tag{90}$$

Integrating Eq. (89) and substituting the resultant into Eq. (79) yield

$$\lambda = -\zeta\omega_1 - \frac{\mu}{4\omega_1}\lim_{t\to\infty}\frac{1}{t}\int_0^{\mathrm{t}}\cos\Gamma(t)\,\mathrm{d}t. \tag{91}$$

It can be shown that $\Gamma(t)$ is ergodic. So, the time average in Eq. (91) can be replaced by ensemble average. Thus,

$$\lambda = -\zeta\omega_1 - \frac{\mu}{4\omega_1}E[\cos\Gamma]. \tag{92}$$

The stationary probability density $p(\gamma)$ for evaluating $E[\cos\Gamma]$ is obtained from solving the following reduced FPK equation associated with the second equation of Eq. (87):

$$\sigma^2\frac{\mathrm{d}^2p}{\mathrm{d}\gamma^2} - 2\frac{\mathrm{d}}{\mathrm{d}\gamma}\left[\left(\Omega - 2\omega_1 + \frac{\mu}{2\omega_1}\sin\gamma\right)p\right] = 0. \tag{93}$$

The solution of Eq. (93) satisfying periodic condition is

$$p(\gamma) = C\exp\left\{\frac{2}{\sigma^2}\left[(\Omega - 2\omega_1)\gamma - \frac{\mu}{2\omega_1}\cos\gamma\right]\right\}\int_\gamma^{2\pi+\gamma}\exp\left\{-\frac{2}{\sigma^2}\left[(\Omega - 2\omega_1)\gamma - \frac{\mu}{2\omega_1}\cos\gamma\right]\right\}\mathrm{d}\,\gamma, \tag{94}$$

where C is a normalizing constant.

For optimally controlled system (79), the fully completed averaged Itô equations are

$$\begin{aligned}
\mathrm{d}A &= \left[\left(-\xi\omega_1 A - \frac{\mu A}{4\omega_1}\cos\Gamma\right) - (V_{02} + V_{12}^c\cos\Gamma + V_{12}^s\sin\Gamma)\frac{A}{2R_1\omega_1^2}\right]\mathrm{d}t,\\
\mathrm{d}\Gamma &= \left[\left(\Omega - 2\omega_1 + \frac{\mu}{2\omega_1}\sin\Gamma\right) - (-V_{12}^c\sin\Gamma + V_{12}^s\cos\Gamma)\frac{1}{R_1\omega_1^2}\right]\mathrm{d}t + \sigma\,\mathrm{d}B(t).
\end{aligned} \tag{95}$$

Following a derivation similar to that from Eqs. (87) to (94), we obtain the stationary probability density

$$\begin{aligned}
p^c(\gamma) = {} & C\exp\left\{\frac{2}{\sigma^2}\left[(\Omega - 2\omega_1)\gamma - \frac{\mu}{2\omega_1}\cos\gamma - \frac{1}{R_1\omega_1^2}\left(V_{12}^c\cos\gamma + V_{12}^s\sin\gamma\right)\right]\right\}\\
& \times\int_\gamma^{2\pi+\gamma}\exp\left\{\frac{-2}{\sigma^2}\left[(\Omega - 2\omega_1)\gamma - \frac{\mu}{2\omega_1}\cos\gamma - \frac{1}{R_1\omega_1^2}(V_{12}^c\cos\gamma + V_{12}^s\sin\gamma)\right]\right\}\mathrm{d}\gamma,
\end{aligned} \tag{96}$$

and Lyapunov exponent

$$\lambda^c = -\xi\omega_1 - \frac{V_{02}}{2R_1\omega_1^2} - \left(\frac{\mu_1}{4\omega_1} + \frac{V_{12}^c}{2R_1\omega_1^2}\right)E[\cos\Gamma] - \frac{V_{12}^s}{2R_1\omega_1^2}E[\sin\Gamma] \tag{97}$$

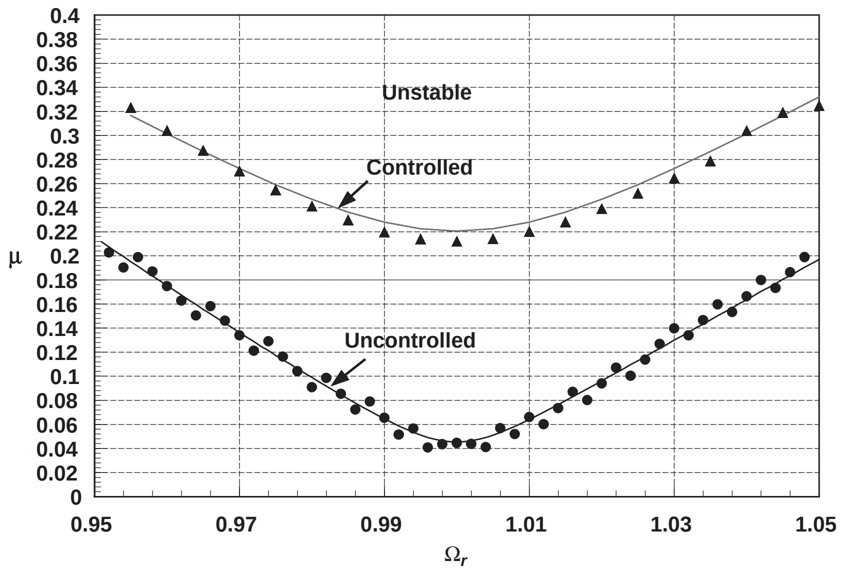

Fig. 10. Primary unstable region of linear system (79). $\omega_1 = 1.0$, $\zeta = 0.01$, $\Omega_r = 2\omega_1/\Omega$, $\sigma^2 = 0.01$, $L_{102} = 0.998$, $L_{104} = 0$, $L^c_{11} = L^s_{11} = 0$, $\eta = 1.0$, $R_1 = 1.0$. ——, analytical result; ▲●, from digital simulation.

$\lambda = 0$ or $\lambda^c = 0$ yields the boundary of the regions of stable and unstable with probability one in parameter space. A numerical result is obtained and shown in Fig. 10. The result obtained from digital simulation is also shown for comparison. It is seen that the two results are in very good agreement and the unstable region can be reduced remarkably by the feedback control. Thus, it is always possible to suppress the cable vibration due to parametric excitation of bounded noise via feedback stabilization.

6. Concluding remarks

In the present paper, a stochastic averaging method for s.d.o.f. strongly non-linear oscillators subject to external and/or parametric excitations of bounded noise has been developed. Based on the stochastic averaging method and the stochastic dynamical programming principle, a strategy for non-linear stochastic optimal control of the strongly non-linear systems under bounded noise excitation has been proposed. The control strategy has been applied to the control of a cable under external bounded noise excitation. It has also been applied to stabilizing a cable under parametric bounded noise excitation. All the analytical results are confirmed with those from digital simulation. It has been shown that the stochastic averaging method developed is quite accurate and the control strategy proposed is quite effective and efficient in reducing the vibration amplitude and in extending the stability region.

Although the application of the proposed strategy to the control of cable vibration and in stability has been illustrated with its first mode, it is easily applied to other cable mode by changing modal parameters. If it is detected that the cable mode in resonance changes, only the modal parameters needed to be changed.

It should be noted that the control strategy proposed in the present paper is approximate optimal but not exact optimal since the dynamical programming principle is applied to the averaged control system rather that the original control system. The averaged system is the first approximation of the original system. It would be better to call the control as quasi-optimal one. In the present paper, however, it is called optimal control just for simplicity.

Acknowledgements

The work reported in this paper was supported by the National Natural Science Foundation of China under a key grant and Grant Nos. 19972059, 10002015 and 60034010, Special Fund for Doctor Programs in Institutions of Higher Learning of China under Grant No. 20020335092 and Zhejiang Provincial Natural Science Foundation of China under Grant No.102040.

Appendix A. Solution of final dynamical programming equation

For ergodic control of cable vibration, the final dynamical programming equation is of the form

$$\eta = L_1(a,\gamma) + m_1(a,\gamma)\frac{\partial V}{\partial a} + m_2(a,\gamma)\frac{\partial V}{\partial \gamma} - \left(\frac{m_{11}(a)}{4R_1} + \frac{m_{21}(a)}{4R_2}\right)\left(\frac{\partial V}{\partial a}\right)^2 - \left(\frac{m_{12}(a)}{4R_1} + \frac{m_{22}(a)}{4R_2}\right)\left(\frac{\partial V}{\partial \gamma}\right)^2 + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial \gamma^2}. \tag{A.1}$$

Suppose that $L_1(a,\gamma)$ can be expanded into Fourier series with respect to γ,

$$L_1(a,\gamma) = L_{10}(a) + \sum_{i=1}^{\infty} (L_{1i}^c(a)\cos i\gamma + L_{1i}^s(a)\sin i\gamma). \tag{A.2}$$

Then the solution of Eq. (A.1) can be assumed of the form of the following Fourier series:

$$V(a,\gamma) = V_0(a) + \sum_{i=1}^{\infty} (V_i^c(a)\cos i\gamma + V_i^s(a)\sin i\gamma). \tag{A.3}$$

Note that coefficients m_1 and m_2 in Eqs. (60) are of the form

$$\begin{aligned} m_1(a,\gamma) &= m_{10}(a) + m_{11}^c(a)\cos\gamma, \\ m_2(a,\gamma) &= m_{20}(a) + m_{21}^s(a)\sin\gamma \end{aligned} \tag{A.4}$$

and m_{ij} $(i,j=1,2)$ are independent of γ. Substituting Eqs. (A.2)–(A.4) into Eq. (A.1), we obtain the following series of equation:

$$\begin{aligned}\eta &= L_{10}(a) + m_{10}(a)\frac{\mathrm{d}V_0}{\mathrm{d}a} + \frac{1}{2}\frac{\mathrm{d}V_1^c}{\mathrm{d}a}m_{11}^c \\ &\quad - \frac{1}{2}V_1^c m_{21}^s - \left\{\left(\frac{\mathrm{d}V_0}{\mathrm{d}a}\right)^2 + \frac{1}{2}\sum_{i=1}^{\infty}\left[\left(\frac{\mathrm{d}V_i^c}{\mathrm{d}a}\right)^2 + \left(\frac{\mathrm{d}V_i^s}{\mathrm{d}a}\right)^2\right]\right\} \\ &\quad \times \left(\frac{m_{11}(a)}{4R_1} + \frac{m_{21}(a)}{4R_2}\right) - \frac{1}{2}\left[\sum_{i=1}^{\infty} i^2\left(V_i^{c^2} + V_i^{s^2}\right)\right]\left(\frac{m_{12}(a)}{4R_1} + \frac{m_{22}(a)}{4R_2}\right),\end{aligned} \tag{A.5}$$

$$\begin{aligned}0 &= L_{11}^c(a) + \frac{\mathrm{d}V_0}{\mathrm{d}a}m_{11}^c(a) + \frac{1}{2}\frac{\mathrm{d}V_2^c}{\mathrm{d}a}m_{11}^c(a) - V_2^s m_{21}^s(a) + \frac{\mathrm{d}V_1^c}{\mathrm{d}a}m_{10}(a) + V_1^s m_{20}(a) - \frac{\sigma^2}{2}V_1^c \\ &\quad - \left[2\frac{\mathrm{d}V_0}{\mathrm{d}a}\frac{\mathrm{d}V_1^c}{\mathrm{d}a} + \sum_{i=1}^{\infty}\left(\frac{\mathrm{d}V_i^c}{\mathrm{d}a}\frac{\mathrm{d}V_{i+1}^c}{\mathrm{d}a} + \frac{\mathrm{d}V_i^s}{\mathrm{d}a}\frac{\mathrm{d}V_{i+1}^s}{\mathrm{d}a}\right)\right]\left(\frac{m_{11}(a)}{4R_1} + \frac{m_{21}(a)}{4R_2}\right) \\ &\quad - \left[\sum_{i=1}^{\infty} i(i+1)\left(V_i^c V_{i+1}^c + V_i^s V_{i+1}^s\right)\right]\left(\frac{m_{12}(a)}{4R_1} + \frac{m_{22}(a)}{4R_2}\right),\end{aligned} \tag{A.6}$$

$$\begin{aligned}0 &= L_{11}^s(a) + \frac{\mathrm{d}V_1^s}{\mathrm{d}a}m_{10}(a) + \frac{1}{2}\frac{\mathrm{d}V_2^s}{\mathrm{d}a}m_{11}^c(a) + V_2^c m_{21}^s(a) - V_1^c m_{20}(a) - \frac{\sigma^2}{2}V_1^s \\ &\quad - \left[2\frac{\mathrm{d}V_0}{\mathrm{d}a}\frac{\mathrm{d}V_1^s}{\mathrm{d}a} + \sum_{i=1}^{\infty}\left(\frac{\mathrm{d}V_i^c}{\mathrm{d}a}\frac{\mathrm{d}V_{i+1}^s}{\mathrm{d}a} - \frac{\mathrm{d}V_i^s}{\mathrm{d}a}\frac{\mathrm{d}V_{i+1}^c}{\mathrm{d}a}\right)\right]\left(\frac{m_{11}(a)}{4R_1} + \frac{m_{21}(a)}{4R_2}\right) \\ &\quad + \left[\sum_{i=1}^{\infty} i(i+1)\left(V_i^s V_{i+1}^c - V_i^c V_{i+1}^s\right)\right]\left(\frac{m_{12}(a)}{4R_1} + \frac{m_{22}(a)}{4R_2}\right),\end{aligned} \tag{A.7}$$

$$\cdots\cdot$$

The higher harmonic terms in the solution can be neglected. An approximate solution of Eq. (A.1) is of the form

$$V(a,\gamma) \approx V_0(a) + V_1^c(a)\cos\gamma + V_1^s(a)\sin\gamma. \tag{A.8}$$

To evaluate $V_0(a)$, $V_1^c(a)$ and $V_1^s(a)$ from the coupled ordinary differential equations (A.5)–(A.7) the Fourier coefficients are further expanded into Taylor series:

$$\begin{aligned}m_{10}(a) &= m_{101}a + m_{103}a^3 + \cdots, \\ m_{20}(a) &= m_{200} + m_{202}a^2 + \cdots, \\ m_{11}^c(a) &= m_{111}^c + m_{113}^c a^2 + \cdots,\end{aligned}$$

$$
\begin{aligned}
m^s_{21}(a) &= m^s_{210}a^{-1} + m^s_{212}a + \cdots,\\
m_{11}(a) &= m_{110} + m_{112}a^2 + \cdots,\\
m_{12}(a) &= m_{120}a^{-2} + m_{122} + \cdots,\\
m_{21}(a) &= m_{212}a^2 + m_{214}a^4 + \cdots,\\
m_{22}(a) &= m_{220} + m_{222}a^2 + \cdots,
\end{aligned}
\tag{A.9}
$$

$$
\begin{aligned}
L_{10}(a) &= L_{100} + L_{102}a^2 + L_{104}a^4 + \cdots,\\
L^c_{11}(a) &= L^c_{111}a + L^c_{113}a^3 + \cdots,\\
L^s_{11}(a) &= L^s_{111}a + L^s_{113}a^3 + \cdots,
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
V_0(a) &= V_{02}a^2 + V_{04}a^4 + \cdots,\\
V^c_1(a) &= V^c_{11}a + V^c_{13}a^3 + \cdots,\\
V^s_1(a) &= V^s_{11}a + V^s_{13}a^3 + \cdots.
\end{aligned}
\tag{A.11}
$$

Substituting Eqs. (A.9)–(A.10) into Eqs. (A.4)–(A.6) and letting the coefficients of the same power of a vanish yield the coefficients of polynomials $V_0(a)$, $V_1(a)$ and $V^s_1(a)$.

For controlled system (57) and cost function $L(a,\gamma,\mathbf{u}) = L_1(a,\gamma) + \mathbf{u}^{\mathrm{T}}\mathbf{Ru}$, where $L_1(a,\gamma) = L_{100} + L_{102}a^2 + L^c_{111}a\cos\gamma + L^s_{111}a\sin\gamma$, the approximate solution (A.8) is of the form of Eq. (64). For linearized controlled system (79), the approximate solution is

$$
V(a,\gamma) = V_{02}a^2 + V^c_{12}a^2\cos\gamma + V^s_{12}a^2\sin\gamma. \tag{A.12}
$$

References

[1] G.W. Housner, L.A. Bergman, T.K. Caugley, A.G. Chassiakos, R.O. Claus, S.F. Masri, R.E. Skelton, T.T. Soong, B.F. Spencer Jr., J.P.T. Yao, Structural control: past, present, and future, *Journal of Engineering Mechanics, American Society of Civil Engineers* 123 (9) (1997) 897–971.

[2] W.Q. Zhu, Z.G. Ying, Optimal nonlinear feedback control of quasi Hamiltonian systems, *Science in China, Series A* 42 (1999) 1213–1219.

[3] W.Q. Zhu, Z.G. Ying, Y.Q. Ni, J.M. Ko, Optimal nonlinear stochastic control of hysterctic systems, *Journal of Engineering Mechanics, American Society of Civil Engineers* 126 (2000) 1027–1032.

[4] W.Q. Zhu, Z.G. Ying, T.T. Soong, An optimal nonlinear feedback control strategy for randomly excited structural systems, *Nonlinear Dynamics* 24 (2001) 31–51.

[5] W.Q. Zhu, Y.Q. Yang, Stochastic averaging of quasi-non-integrable-Hamiltonian systems, *Journal of Applied Mechanics, American Society of Mechanical Engineers* 64 (1997) 157–164.

[6] W.Q. Zhu, Z.L. Huang, Y.Q. Yang, Stochastic averaging of quasi integrable Hamiltonian systems, *Journal of Applied Mechanics, American Society of Mechanical Engineers* 64 (1997) 975–984.

[7] W.Q. Zhu, Z.L. Huang, Y. Suzuki, Stochastic averaging and Lyapunov exponent of quasi partially integrable Hamiltonian systems, *International Journal of Nonlinear Mechanics* 37 (2002) 419–437.

[8] W.H. Fleming, R.W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975.

[9] W.H. Fleming, H.M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, 1993.

[10] J.M. Yong, X.Y. Zhou, *Stochastic Control, Hamiltonian Systems and HJB Equations*, Springer, New York, 1999.

[11] W.Q. Zhu, Z.L. Huang, Feedback stabilization of quasi integrable Hamiltonian systems, *Journal of Applied Mechanics, American Society of Mechanical Engineers* 70 (1) (2003) 129–136.

[12] W.Q. Zhu, Z.L. Huang, M.L. Deng, Feedback minimization of first-passage failure of quasi non-integrable Hamiltonian systems, *International Journal of Nonlinear Mechanics* 37 (6) (2002) 1057–1071.
[13] W.Q. Zhu, Z.L. Huang, M.L. Deng, First-passage failure and its feedback minimization of quasi partially integrable Hamiltonian systems, *International Journal of Nonlinear Mechanics* 38 (8) (2003) 1133–1148.
[14] W.Q. Zhu, Z.G. Ying, Nonlinear stochastic optimal control of partially observable linear structures, *Engineering Structures* 24 (2002) 333–342.
[15] R.L. Stratonovich, *Topics in the Theory of Random Noise*, Vol. II, Gorden and Breach, New York, 1967.
[16] M.F. Dimentberg, *Statistical Dynamics of Nonlinear and Time Varying System*, Research Studies Press, Taunton, 1988.
[17] Y.K. Lin, Q.C. Li, T.C. Su, Application of a new wind turbulence model in predicting motion stability of wind-excited long-span bridges, *Journal of Wind Engineering and Industrial Aerodynamics* 49 (1993) 507–516.
[18] S.T. Ariaratnam, Stochastic stability of viscoelastic systems under bounded noise excitation, in: A. Naess, S. Krenk (Eds.), *Proceedings of the IUTAM Symposium*, Kluwer Academic Publishers, Dordrecht, 1996, pp. 11–18.
[19] W.Y. Liu, W.Q. Zhu, Z.L. Huang, Effect of bounded noise on chaotic motion of Duffing oscillator under parametric excitation, *Chaos, Solitons and Fractals* 12 (2001) 527–537.
[20] G.Q. Cai, Y.K. Lin, Random vibration of strongly nonlinear systems, *Nonlinear Dynamics* 24 (2001) 3–15.
[21] W.Q. Zhu, Z.L. Huang, Response and stability of strongly non-linear oscillators under wide-band random excitation, *International Journal of Nonlinear Mechanics* 36 (2001) 1235–1250.
[22] Z.L. Huang, W.Q. Zhu, Stochastic averaging of strongly non-linear oscillator under combined harmonic and white noise excitations, *Journal of Sound and Vibration* 238 (2) (2000) 233–256.
[23] Y.K. Lin, G.Q. Cai, *Probabilistic Structural Dynamics, Advanced Theory and Applications*, McGraw-Hill, New York, 1995.
[24] Y. Fujino, P. Warnitchai, B.M. Pacheco, Active stiffness control of cable vibration, *Journal of Applied Mechanics, American Society of Mechanical Engineers* 60 (1993) 948–953.
[25] Y. Achkire, A. Preumont, Active tendon control of cable-stayed bridges, *Earthquake Engineering and Structural Dynamics* 25 (1996) 585–597.
[26] R.W. Gehle, S.F. Masri, Active control of shallow, slack cable using the parametric control of end tension, *Nonlinear Dynamics* 17 (1998) 77–94.
[27] Y.L. Xu, Z. Yu, Vibration of inclined sag cables with oil dampers in cable stayed bridges, *Journal of Bridge Engineering* 3 (4) (1998) 194–203.
[28] S. Krenk, Vibrations of a taut cable with an external damper, *Journal of Applied Mechanics, American Society of Mechanical Engineers* 67 (2000) 772–776.
[29] B.N. Sun, Z.G. Wang, J.M. Ko, Y.Q. Ni, Cable oscillation induced by parametric excitation in cable-stayed bridges, in: J.M. Ko, Y.L. Xu (Eds.), *Advances in Structural Dynamics*, Vol. 1, Elsevier, Amsterdam, 2000, pp. 553–560.
[30] J.L. Lilien, A. Pint da Costa, Vibration amplitude caused by parametric excitation of cable stayed structures, *Journal of Sound and Vibration* 174 (1994) 69–90.
[31] A. Pint da Costa, J.A.C. Martins, F. Branco, J.L. Lilien, Oscillations of bridge stay cables induced by periodic motion of deck and/or towers, *Journal of Engineering Mechanics, American Society of Civil Engineers* 122 (1996) 613–622.
[32] M. Virlogeux, Cable vibration in cable-stayed bridges, in: A. Larsen, S. Esdahl (Eds.), *Bridge Aerodynamics*, Balkema, Rotterdam, 1998, pp. 213–233.
[33] H. Yamaguchi, Y. Fujino, Stayed cable dynamics and its vibration control, in: A. Larsen, S. Esdahl (Eds.), *Bridge Aerodynamics*, Balkema, Rotterdam, 1998, pp. 235–253.